An Adjoint Variable Method for Structural Design Sensitivity Analysis of a Distinct Eigenvalue Problem

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New adjoint variable method for design sensitivity analysis of distinct eigenvlaues and eigenvectors is presented. In the viewpoint of efficiency for the design sensitivity analysis of eigenvectors especially, the developed adjoint variable method is required to compute adjoint variables from simultaneous linear equations, the so-called adjoint equations, instead of linear combination of eigenvectors. Once we obtain the adjoint variables, design sensitivity analysis of response function that is given in terms of eigenvalues, eigenvectors and design variables can be computed directly. In this way, design sensitivity analysis of eigenvectors can be obtained by using eigenvalues and their corresponding eigenvectors of the mode being differentiated only. To verify the proposed method, numerical examples are demonstrated. This can have considerable impact on computer implementation of the developed method in the design sensitivity analysis of eigenproblem needed for practical applications.

Key Words : Design Sensitivity Analysis, Adjoint Variable Method, Design Optimization, Eigenproblem, Modal Analysis

1. Introduction

Eigenproblems are commonly considered in structural stability, buckling, noise and vibration analyses. Design sensitivity analysis of an eigenproblem computes the rate of changes of response-dependent function, for instance, eigenvalues and eigenvectors, with respect to the perturbation of design variables. Design sensitivity analysis is also an essential step to systematically improve the existing design and to optimize a system with the aid of gradient-based optimization techniques (Haftka and Adelman 1989; Haug, et al 1986).

Fox and Kapoor (1968) developed general technique to compute design sensitivity of eigenvalues for symmetric matrices. However this method requires all eigenvalues and eigenvectors for the system, which is computationally expensive for a large-scale problems. Plaut and Hussyin (1973) and Rudisill(1974) developed formulas for second-order design sensitivity analysis of eigenvalues.

To analyze the design sensitivity of the eigenvectors, we can use direct differentiation method that differentiates an eigenproblem with respect to design variables and solves the simultaneous equation directly for the eigenvector derivatives (Haug, et at 1986). However, since the simultaneous equation is singular, an orthonormality condition is adapted in the solution process (Jung, et al 1997; Lee and Jung 1997). Nelson introduced a normalization condition where the largest component of the eigenvector is unity (Nelson 1976). Note that Nelson and Lee and Jung methods require only the eigenvalues and eigenvectors for the modes being differentiated.

Design sensitivity analysis of eigenvectors can also be obtained by the modal method (Fox and Kapoor 1968; Rogers 1970) and a modified modal method (Wang 1991; Lin, Lim and Wang 1997), whereby the design derivatives of the eigenvectors are expanded in terms of the eigenvectors. The modal method approximates the derivatives of eigenvectors as a linear combi-

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nation of eigenvectors. This method can be computationally expensive and impractical if large number of eigenvectors is needed to accurately represent the derivatives of eigenvectors. The modified modal method is developed to reduce the number of eigenvectors needed to represent the derivatives by including an additional term in the linear combination of eigenvectors, where inaccuracy of the approximation has been leaded for need of relatively higher modes. Review and comparison of several methods for design sensitivity analysis of eigenvectors have been carried out (Stutter, et al 1988). Note that the modal methods compute the design sensitivity of state variables instead of the design sensitivity of response functions.

Proposed new method for calculating derivatives of eigenvalues and eigenvectors is purely adjoint variable method that requires evaluation of the adjoint variables from the simultaneous system equation, the so-called adjoint equation. It is very important to note that the adjoint equation requires only the eigenvalues and associated eigenvectors of the modes being differentiated. Once we obtain adjoint variables, we can evaluate design sensitivity coefficients of response function directly. Therefore, when the dimensions of design variables are larger than the number of response functions, the developed method is generally more efficient than the direct differentiation method and the modal methods. Numerical examples are given to verify the developed method,Further, the developed method can be easily implemented into a commercial finite element program to carry out the design sensitivity analysis of eigenproblems needed for practical applications.

2. Definition of Eigenproblem

Undamped free vibration and linear buckling analysis lead to the generalized eigenproblem as follows:

$$\mathrm{K}\mathrm{u}_i=\lambda_i\mathrm{M}\mathrm{u}_i \tag{1}$$

where K represents the stiffness matrix, M represents the mass matrix in vibration analysis or geometric stiffness matrix in buckling analysis. The eigenvalue λ_i and associated eigenvectors u_i represent the i-th free vibration frequency squared and corresponding mode shape vectors, respectively. In care of buckling problems, the lowest eigenvalue $\lambda_1\equiv\lambda_{cr}$ is associated with buckling load.

Since the mode shape is often normalized with a symmetric positive definite matrix, we take M -orthonormality condition as follows:

$$\mathrm{u}_i^T\mathrm{M}\mathrm{u}_j=\delta_{ij} \tag{2}$$

where δ_{ij} represents the Kronecker delta and the right superscript T denotes the transposition of a matrix.

3. Design Sensitivity Analysis of Eigenproblem

Consider a general response function of an eigenproblem represented in terms of eigenvalues, eigenvectors, and a design variable as follows:

$$g=g(\lambda_i,\ \mathrm{u}_i,\ b) \tag{3}$$

where b denotes design variable. It is assumed that the response function whose design sensitivity needs to be evaluated is continuous and differentiable with respect to its arguments.

To develop the adjoint variable method with the aid of the variational principle for design sensitivity analysis (Arora and Cardoso 1992), we first define the augmented function as

$$A=g(\lambda_i,\ \mathrm{u}_i,\ b)+\mathrm{z}^T(\mathrm{K}-\lambda_i\mathrm{M})\mathrm{u}_i+y\left(\frac{1}{2}-\frac{1}{2}\mathrm{u}_i^T\mathrm{M}\mathrm{u}_i\right) \tag{4}$$

where y and z are the adjoint variables, the so-called Lagrange multipliers, for eigenvalues and eigenvectors, respectively, which will be determined later.

According to the variational principle for design sensitivity analysis, the total design variation of the response function can be represented as explicit design variation of the augmented function given in Eq. (4), i. e.,

$$\frac{dg}{db}=\frac{\partial A}{\partial b}=\frac{\partial g}{\partial b}+\mathrm{z}^T\left[\frac{\partial \mathrm{K}}{\partial b}-\lambda_i\frac{\partial \mathrm{M}}{\partial b}\right]\mathrm{u}_i$$

$$-\frac{1}{2}y\mathrm{u}_i^T\frac{\partial \mathrm{M}}{\partial b}\mathrm{u}_i \tag{5}$$

Now the adjoint equation can be obtained by requiring the implicit design variation of the augmented function in Eq. (4) to vanish. This leads to the following equation:

$$0=\frac{\partial g}{\partial \lambda_i}\frac{d\lambda}{db}+\frac{\partial g}{\partial \mathrm{u}_i}\cdot\frac{d\mathrm{u}_i}{db}-\frac{d\lambda_i}{db}\mathrm{z}^T(\mathrm{K}-\lambda_i\mathrm{M})\cdot\frac{d\mathrm{u}_i}{db}-y\mathrm{u}_i^T\mathrm{M}\frac{d\mathrm{u}_i}{db}\ \text{for}\ \forall\ \frac{d\lambda_i}{db},\ \frac{d\mathrm{u}_i}{db} \tag{6}$$

where the operator '$\cdot$' denotes the inner product. Rearranging Eq. (6), we have

$$0=\frac{d\lambda_i}{db}\left[\frac{\partial g}{\partial \lambda_i}-\mathrm{u}_i^T\mathrm{M}\mathrm{z}\right]+\frac{d\mathrm{u}_i}{db}\cdot\left[\frac{\partial g}{\partial \mathrm{u}_i}+(\mathrm{K}-\lambda_i\mathrm{M})\mathrm{z}-y\mathrm{M}\mathrm{u}_i\right]\ \text{for}\ \forall\ \frac{d\lambda_i}{db},\ \frac{d\mathrm{u}_i}{db} \tag{7}$$

where the fact that K and M are symmetric matrices is employed. Since Eq. (7) holds for arbitrary $d\lambda_i/db$ and $d\mathrm{u}_i/db$, we have the adjoint equations as follows:

$$\begin{bmatrix}\mathrm{K}-\lambda_i\mathrm{M} & -\mathrm{M}\mathrm{u}_i\\ -\mathrm{u}_i^T\mathrm{M} & 0\end{bmatrix}\begin{bmatrix}\mathrm{z}\\ y\end{bmatrix}=\begin{bmatrix}-\dfrac{\partial g}{\partial \mathrm{u}_i}\\ -\dfrac{\partial g}{\partial \lambda_i}\end{bmatrix} \tag{8}$$

It can be observed from Eq. (8) that the coefficient matrix is nonsingular if the eigenvlaues are distinct and composed of the stiffness and mass matrices and the eigenvalue and eigenvectors of the i-th mode. The right-hand side of Eq. (8) can be given explicitly from the response function. Therefore, it is very important to note that the adjoint variables for the given response function can be evaluated by using the eigenvalues and corresponding eigenvectors of the mode being differentiated only. Once we solve the system equation of Eq. (8), we substitute the adjoint variables into Eq. (5) to obtain the design sensitivity coefficients of the response function. Moreover, when the dimensions of design variables are larger than the number of response functions, the developed method is generally more efficient than the modal methods (Haug, et al 1986). However, it must be pointed out that there are more design variables comparing with the number of response functions in eigenproblems, in general.

4. Examples

Two numerical examples are illustrated here to calculate the design sensitivity coefficients of the eigenproblems and compare with those evaluated by using exact differentiation or the central finite difference method. Throughout the examples, eigenvalues, eigenvectors and system equations are computed by MATLAB (1994).

4.1 Undamped four-degree-of-freedom system

Consider the design sensitivity of the first eigenvalue and its corresponding eigenvectors of an undamped four-degree-of-freedom system (see Fig. 1) with mass and stiffness matrices given as follows (Wang 1991):

$$\mathrm{M}=\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{bmatrix}kg\ ;\ \mathrm{K}=\begin{bmatrix}3&-1&0&0\\-1&3&-2&0\\0&-2&7&-5\\0&0&-5&5\end{bmatrix}\times 10^5\ N/m \tag{9}$$

where the material constants are as follows:

$$[m_1,m_2,m_3,m_4]=[1,2,3,4]\ (kg),$$
$$[k_1,k_2,k_3,k_4]=[2,1,2,5]\ (\times 10^5\ N/m)$$

The response functions whose design sensitivities are required are defined as follows:

$$\begin{aligned}g_1&=\lambda_1\\ g_2&=\mathrm{u}_1^T[1\ 0\ 0\ 0]^T\\ g_3&=\mathrm{u}_1^T[0\ 1\ 0\ 0]^T\\ g_4&=\mathrm{u}_1^T[0\ 0\ 1\ 0]^T\\ g_5&=\mathrm{u}_1^T[0\ 0\ 0\ 1]^T\end{aligned} \tag{10}$$

where g_1 is the response function for the first eigenvalue and g_2 through g_5 are response functions for the elements of the first eigenvectors, respectively. Note that λ_1 and u_1 represent the first eigenvalue and the first eigenvector for the given

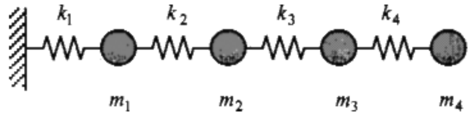

Fig. 1 Undamped four-degree-of-freedom system.

system, respectively.

We now consider the design sensitivity analyses of the first eigenvalue and eigenvector for the following two cases:

Case 1 : $b=m_1$, i. e.,

$$\frac{\partial \mathrm{M}}{\partial b}=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\quad \frac{\partial \mathrm{K}}{\partial b}=0 \tag{11}$$

Case 2 : $b=k_3$, i. e.,

$$\frac{\partial \mathrm{M}}{\partial b}=0,\quad \frac{\partial \mathrm{K}}{\partial b}=\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}\times 10^5 \tag{12}$$

To compute the design sensitivity of the first eigenvalue, we first solve the adjoint equation as follows:

$$\begin{bmatrix}\mathrm{K}-\lambda_1\mathrm{M} & -\mathrm{Mu}_1\\ -\mathrm{u}_1^T\mathrm{M} & 0\end{bmatrix}\begin{bmatrix}\mathrm{z}_1\\ y_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\-1\end{bmatrix} \tag{13}$$

Then using Eq. (5), the design sensitivity for the first eigenvalue is given for two cases as follows:

Case 1:

$$\frac{dg_1}{db}=-\lambda_1\mathrm{z}_1^T\frac{\partial \mathrm{M}}{\partial b}\mathrm{u}_1-\frac{1}{2}y_1\mathrm{u}_1^T\frac{\partial \mathrm{M}}{\partial b}\mathrm{u}_1 \tag{14}$$

Case 2:

$$\frac{dg_1}{db}=\mathrm{z}_1^T\frac{\partial \mathrm{K}}{\partial b}\mathrm{u}_1 \tag{15}$$

To evaluate the design sensitivity for the eigenvectors, for instant g_2, adjoint equation and sensitivity expressions become as follows:

$$\begin{bmatrix}\mathrm{K}-\lambda_1\mathrm{M} & -\mathrm{Mu}_1\\ -\mathrm{u}_1^T\mathrm{M} & 0\end{bmatrix}\begin{bmatrix}\mathrm{z}_2\\ y_2\end{bmatrix}=\begin{bmatrix}-1\\0\\0\\0\\0\end{bmatrix} \tag{16}$$

Case 1:

$$\frac{dg_2}{db}=-\lambda_1\mathrm{z}_2^T\frac{\partial \mathrm{M}}{\partial b}\mathrm{u}_1-\frac{1}{2}y_2\mathrm{u}_1^T\frac{\partial \mathrm{M}}{\partial b}\mathrm{u}_1 \tag{17}$$

Case 2:

$$\frac{dg_2}{db}=\mathrm{z}_2^T\frac{\partial \mathrm{K}}{\partial b}\mathrm{u}_1 \tag{18}$$

Note that the first eigenvalue and its corresponding eigenvectors are only employed to evaluate the design sensitivity coefficient of the first element of the first eigenvector. Similarly we can compute the design sensitivity for the other elements of the first eigenvector.

The design sensitivity coefficients of the first eigenvalue and eigenvector for the two cases are given in Table 1. These results show very good agreement with the exact derivatives. It is important to note that the first eigenvalue and corresponding eigenvector are only used for design sensitivity analysis of the first eigenvalue and eigenvector in the developed adjoint variable method.

4.2 Cantilever beam

Consider the design sensitivity analysis for eigenvlaues and eigenvectors of a uniform-thick-

Table 1 comparison of design sensitivity coefficients of the first eigenvalue and eigenvectors with exact derivatives

	Case 1		Case 2	
	Exact	AVM	Exact	AVM
$d\lambda_1/db$	-48.622780	-48.622780	520.823568	520.823568
$\frac{d\mathrm{u}_1}{db}$	0.001617	0.001617	0.008497	0.008497
	-0.000565	-0.000565	0.024517	0.024517
	-0.001493	-0.001493	-0.006397	-0.006397
	-0.001712	-0.001712	-0.005160	-0.005160

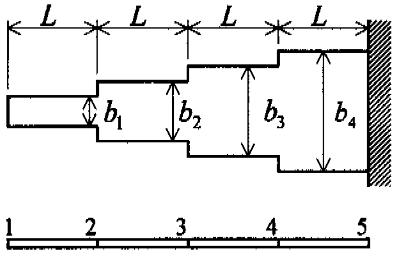

Fig. 2 Cantilever beam with variable widths and its finite element model.

Table 2 First three eigenvlaues and eigenvectors of the cantilever beam

	Mode 1	Mode 2	Mode 3
λ_i	4.361e2	8.613e3	5.414e4
	4.4857	-5.3462	5.4658
	-3.2870	11.1978	-20.1031
	2.8643	-0.1909	-2.2890
u_i	-3.1137	7.8790	-4.1895
	1.4228	2.1465	-0.2207
	-2.5229	0.7027	8.9563
	0.3937	1.2395	1.9728
	-1.4587	-3.4880	-2.5454

ness-cantilever beam with variable widths as shown in Fig. 2. Since the beam is divided into four finite elements, there are eight degrees of freedom for this finite element model, where axial motion is assumed to be negligible. The design variables are chosen three widths of the beam, i. e., $b=[b_1\ b_2\ b_3]^T$.

The material constants for the cantilever beam are as follows:

$$E=200\times10^9\ N/m^2,\ \rho=7800 kg/m^3$$

The geometric properties are taken to be $L=0.5\ m$, $b_1=0.01\ m$, $b_2=0.02\ m$, $b_3=0.03\ m$, and $b_4=0.04\ m$, respectively, and the thickness of the beam is taken to be $t=0.001 m$. It is assumed that the node 5 is fixed. Table 2 shows eigenvalues and eigenvectors for the first three modes.

The design sensitivity analysis of eigenvalues and eigenvectors of the cantilever beam are car-

Table 3 Design sensitivity coefficients of eigenvalues and their comparison with central finite difference method

	AVM§	R(%)
$\lambda_{1,2}$*	-230.8478	100.00
$\lambda_{1,2}$	-66.0417	100.00
$\lambda_{1,3}$	25.3517	100.01
$\lambda_{2,1}$	-2485.9099	100.01
$\lambda_{2,2}$	1064.0475	100.00
$\lambda_{2,3}$	-542.4464	100.00
$\lambda_{3,1}$	-556.3837	99.97
$\lambda_{3,2}$	-886.1520	100.01
$\lambda_{3,3}$	50.8896	99.93

* $\lambda_{i,j}\equiv d\lambda_i/db_j$

§ AVM : Adjoint Variable Method

Table 4 Design sensitivity coefficients of some elements of eigenvectors with respect to design variable b_1 and their comparison with central finite difference method

	AVM	R(%)
$u_{11,1}$	-112.1784	100.00
$u_{14,1}$	64.0150	100.00
$u_{15,1}$	-41.9889	100.00
$u_{18,1}$	45.3606	100.00
$u_{22,1}$	-491.5451	100.01
$u_{23,1}$	110.4510	100.00
$u_{26,1}$	-125.2893	99.99
$u_{27,1}$	-5.9978	99.99
$u_{31,1}$	-14.1984	100.00
$u_{33,1}$	-6.5923	100.00
$u_{36,1}$	17.4019	100.00
$u_{38,1}$	-12.5400	100.01

ried out by using the developed adjoint variable method. The design sensitivity of the eigenvlaues given in Table 3 tells us that reducing the width b_1 will increase the first frequency of the cantilever beam. However increasing the width b_2 will increase the second frequency but decrease the first and third frequencies of the cantilever beam.

Tables 3 and 4 show the design sensitivity results and comparison with those calculated by the central finite difference method. The design perturbation for the finite difference method is chosen as 1% of the design variable. The design sensitivity coefficients of eigenvalues and eigenvectors obtained by applying the developed adjoint variable method are in good agreement with those computed by using the central finite difference method. In the presenting results, we give the ratio R between the design sensitivity coefficients predicted by the adjoint variable method (dg/db) and those by the central finite difference method $(\Delta g/\Delta b)$ as follows:

$$R=\frac{\Delta g/\Delta b}{dg/db}\times 100(\%),$$
$$\Delta g=\frac{1}{2}[g(b+\Delta b)-g(b-\Delta b)] \qquad (19)$$

where g represents a response quantity whose design sensitivity coefficient is calculated. A ratio of 100% means that the predicted design sensitivity coefficient matches exactly with that computed by the finite difference method.

5. Concluding Remarks

Design sensitivity analysis of a distinct eigenvalue problem based on adjoint variable method is presented. Adjoint equation for the response quantity defined in terms of eigenvalues and eigenvectors is obtained by requiring the implicit design variations of the augmented function to vanish. Once we calculate the adjoint variables from the linear systems of equations, we can evaluate the design sensitivity coefficients of the response function. Numerical examples are given to calculate the design sensitivity coefficients of the eigenproblems and verify the accuracy and efficiency of the developed method.

Based on the study on design sensitivity analysis of eigenproblem, we can conclude as follows:

(1) Pure adjoint variable method for design sensitivity analysis of eigenproblem is proposed.

(2) When the dimensions of design variables are larger than the number of response functions, the developed adjoint variable method is more efficient than the modal methods.

(3) The proposed adjoint variable method for design sensitivity of an eigenvector can be computed by using the associated eigenvalue and its corresponding eigenvector instead of linear combination of some eigenvectors.

(4) Design sensitivity analysis of eigenproblems can easily be performed outside of a commercial finite element analysis program.

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